Generalized guidance equation for peaked quantum solitons: the single particle case

Thomas Durt

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Abstract

We study certain non-linear generalisations of the Schrödinger equation which admit static solitonic[2] solutions in absence of external potential acting on the particle. We consider a class of solutions that can be written as a product of a solution of the linear Schrödinger equation with a peaked quantum soliton, in a regime where the size of the soliton is quite smaller than the typical scale of variation of the linear wave. In the non-relativistic limit, the solitons obey a generalized de Broglie-Bohm (dB-B) guidance equation. In first approximation, this guidance equation reduces to the dB-B guidance equation according to which they move at the so-called de Broglie-Bohm velocity along the hydrodynamical flow lines of the linear Schrödinger wave. If we consider a spinorial electronic wave function *à la* Dirac, its barycentre is predicted to move exactly in accordance with the dB-B guidance equation.

1 Introduction

Louis de Broglie proposed in 1927 a realistic interpretation of the quantum theory in which particles are guided by the solution of the linear Schrödinger equation (Ψ_L), in accordance with the so-called guidance equation [16, 17]. The theory was generalised by David Bohm

[1]Aix Marseille Université, CNRS, Centrale Marseille, Institut Fresnel (UMR 7249),13013 Marseille, France.email: thomas.durt@centrale-marseille.fr

[2]By soliton we mean solitary wave.

in 1952 [6, 7], who also applied it to the multiparticle case. Certain ingredients of de Broglie's original idea disappeared in Bohm's formulation, in particular the double solution program, according to which the particle is associated to a wave ϕ distinct from the pilot-wave Ψ_L. This program was never fully achieved, ϕ being sometimes treated as a moving singularity [35], and sometimes as a solution of a non-linear equation [17]. In this paper, we shall consider three non-linear generalisations of Schrödinger's linear equation previously scrutinized in [11], the 1+1 dimensional NLS equation, and the 1+3 dimensional S-N equation in the single [30] and many particles (self-gravitating homogeneous sphere [18]) cases, which are known to admit static solitonic solutions (from now on denoted $e^{-iE_0t/\hbar}\phi^0_{NL}(\mathbf{x})$) in absence of external potential. We aim at finding particular solutions of (19), where an external, linear, potential is present.

Our main idea is to consider an ansatz solution Ψ of (19) which factorizes into the product of two functions Ψ_L and ϕ_{NL} where Ψ_L is a solution of the linear Schrödinger equation. As we shall show Ψ_L can be interpreted as a pilot-wave, while the barycentre of ϕ_{NL} obeys a generalized guidance equation which contains the well-known Madelung-de Broglie-Bohm contribution plus a new contribution due to the internal structure of the soliton. In first approximation and in the appropriate regime to be precised later, Ψ is the product of a stationary soliton-shape solution of the non-linear free equation, denoted ϕ^0_{NL}, moving at de Broglie-Bohm velocity, with the phase of Ψ_L. If we denote A_L and φ_L the amplitude and phase of Ψ_L, through $\Psi_L = A_L \cdot e^{i\varphi_L}$, we find thus

$\Psi(x, y, z, t) \approx e^{-iE_0t/\hbar}\phi^0_{NL}(\mathbf{x} - \int_0^t dt\mathbf{v}_{dB-B})e^{i\varphi_L(t,\mathbf{x})}$,

where

$$\mathbf{v}_{dB-B} = \frac{\hbar}{m}\nabla\varphi_L(x, y, z, t) \tag{1}$$

in accordance with the so-called de Broglie-Bohm (dB-B) guidance equation, and E_0 is the energy of the stationary ground state of the free non-linear equation.

This is true at the lowest order of perturbation only. Actually, the exact solution is predicted to obey

$\Psi(x, y, z, t) \approx \phi'_{NL}(\mathbf{x}, t)e^{i\varphi_L}$,

where as we shall show ϕ'_{NL} is a function of constant L_2 norm ($< \phi'_{NL}|\phi'_{NL} >=$ constant) proportional to ϕ_{NL} ($A_L \cdot \phi_{NL} = \phi'_{NL}$). ϕ'_{NL} obeys a rather complicated non-linear equation and is in general not a solitary wave, by which we mean that its shape varies throughout time.

It also obeys a generalized guidance equation, because the barycentre of ϕ'_{NL} is predicted to move at velocity $\mathbf{v}_{drift}$ $=\mathbf{v}_{dB-B}+\mathbf{v}_{int.}$,

with a structural contribution from the non-linear wave:

$\mathbf{v}_{int.} =< \phi'_{NL}|\frac{\hbar}{im}\bigtriangledown|\phi'_{NL} > / < \phi'_{NL}|\phi'_{NL} >$.

In this approach, where the de Broglie-Bohm point-particle is replaced by a soliton, the wave monism originally proposed by de Broglie is restored. As we shall show, the amplitude A_L of the linear solution may be considered as a computation tool, which disappears at the end of the calculation, to the same size that in the de Broglie-Bohm approach, the linear wave Ψ_L is interpreted as a pilot-field, while the "real" object is the pointlike particle.

From this point of view, the NLS and S-N equations are good candidates for fulfilling the de Broglie double solution program of 1927 [5], in the same sense that Schrödinger equation was a good candidate for realizing de Broglie's wave mechanics program of 1925, provided we take some freedom relatively to de Broglie's original ideas and consider them from a broad perspective. In particular, our factorisability ansatz is incompatible with the superposition principle in the sense that we are not considering solutions of the type Ψ=Ψ_L+ϕ_{NL}. This explains why the tail of the solution $\Psi(x, y, z, t)$ may be arbitrary small nearly everywhere, excepted in a supposedly small region where ϕ_{NL} is strongly peaked. Another non-trivial feature of our model is the appearance of corrections to the dB-B guidance equation. Moreover, the phase matching condition of de Broglie is in general not respected in our approach.

The paper is structured as follows. In section 2 we apply our ansatz for deriving an exact solution of the NLS and S-N equations in absence of external potential. In section 3 we apply a similar scheme in presence of an external potential, but this time the result is no longer exact. However, it is a good approximation of the solution as far as the region where ϕ_{NL} is strongly peaked is sufficiently small. In section 4 we consider a typical scattering experiment and, combining the results of the previous sections, we reproduce an argument originally introduced by de Broglie in order to derive Born's rule. In section 5 we consider a straightforward generalisation of the results derived in the previous sections to the Dirac spinor.

The last section is devoted to discussion and conclusion.

2 No external potential.

2.1 The 1 D NLS equation

The 1+1 dimensional nonlinear Schrödinger equation, here rewritten in dimensionless coordinates, reads

$$i\,\frac{\partial\psi}{\partial s}+\frac{\partial^2\psi}{\partial z^2}+\left|\psi\right|^2\psi=0, \tag{2}$$

We are free to search for stationary solutions

$$\Psi(Z,S)=e^{i\mathcal{E}S}\varphi(Z). \tag{3}$$

The function $\varphi(Z)$ then satisfies the stationary nonlinear Schrödinger equation

$$\frac{\partial^2\varphi}{\partial Z^2}+\left|\varphi\right|^2\varphi=\mathcal{E}\varphi, \tag{4}$$

with $\mathcal{E}\geq 0$. As is well-known, equation (4) has statical and localized solutions of the form

$$\varphi(Z)=\frac{\sqrt{2}\lambda}{\cosh(\lambda Z+\delta)}\,,\qquad \mathcal{E}=\lambda^2, \tag{5}$$

for real parameters λ and δ, which correspond to (bright) static soliton solutions for the nonlinear Schrödinger equation with amplitude $2\mathcal{E}$ (for $|\psi|^2$).

2.2 The 3 D Schrödinger-Newton equation: from single to many particles.

In the single particle case, the so-called Schrödinger-Newton integro-differential equation reads[3] [25]

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t}=-\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m}-Gm^2\int d^3x'\big(\frac{|\Psi(t,\mathbf{x}')|^2}{|\mathbf{x}-x'|}\big)\Psi(t,\mathbf{x}). \tag{6}$$

One can therefore look for a "ground-state" solution to (6) in the form

$$\psi(\mathbf{x},t)=e^{\frac{i\mathcal{E}t}{\hbar}}\varphi(\mathbf{x})\,, \tag{7}$$

[3] This equation is also often referred to as the (attractive) Schrödinger-Poisson equation [8, 24, 4] or the gravitational Schrödinger equation [25].

This leads to a stationary equation for $\varphi(\mathbf{x})$

$$\frac{\hbar^2}{2M}\Delta\varphi(\mathbf{x}) + GM^2\int d^3y\,\frac{\left|\varphi(\mathbf{y})\right|^2}{|\mathbf{x}-\mathbf{y}|}\,\varphi(\mathbf{x}) = \mathcal{E}\varphi(\mathbf{x})\,, \tag{8}$$

which was studied in astrophysics and is known under the name of the *Choquard* equation [28]. In [28], Lieb showed that the energy functional

$$E(\phi) = \frac{\hbar^2}{2M}\int d^3x\,\left|\nabla\phi(\mathbf{x})\right|^2 - \frac{GM^2}{2}\iint d^3x d^3y\,\frac{\left|\phi(\mathbf{y})\right|^2}{|\mathbf{x}-\mathbf{y}|}\left|\phi(\mathbf{x})\right|^2, \tag{9}$$

is minimized by a unique solution $\varphi(\mathbf{x})$ of the Choquard equation (8) for a given L_2 norm $N(\varphi)$. However no analytical expression is known for this ground state. Numerical treatments established that this ground state has a quasi-gaussian shape, and that its size is, in the case $N(\varphi) = 1$, of the order of $\frac{\hbar^2}{GM^3}$.

The generalisation of equation (6) to non-elementary (composite) particles has been tackled by Diósi who considered the problem of a self-interacting sphere in [18], and showed that when the mean width of the center-of-mass wave function is small enough in comparison to the size of the sphere, the self-interaction reduces, in a first approximation, to a non-linear harmonic potential (see [37, 11] for a generalization of Diósi's result):

$-G(\frac{M}{\frac{4\pi R^3}{3}})^2\int_{|\tilde{x}|\leq R,|\tilde{x}'|\leq R} d^3\tilde{x}d^3\tilde{x}'\frac{1}{|\mathbf{x}_{CM}+\tilde{\mathbf{x}}-(\mathbf{x}'_{CM}+\tilde{\mathbf{x}}')|}$

$\approx \frac{GM^2}{R}(-\frac{6}{5}+\frac{1}{2}(\frac{|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|}{R})^2+\mathcal{O}((\frac{|\mathbf{x}_{CM}-\mathbf{x}'_{CM}|}{R})^3))$

Looking for static solutions of the many particles NS equation, one obtains the following equation for the ground state wave function of the center of mass, where $\mathbf{x}$=$\mathbf{x}_{CM}$:

$$\frac{\hbar^2}{2M}\Delta\varphi(\mathbf{x}) - \frac{GM^2}{2R^3}\int d^3y\left|\varphi(\mathbf{y})\right|^2\,|\mathbf{x}-\mathbf{y}|^2\,\varphi(\mathbf{x}) = -\mathcal{E}^D\varphi(\mathbf{x})\,. \tag{10}$$

Here we introduced the effective parameter $\mathcal{E}^D$ as

$$\mathcal{E}^D = \frac{6GM^2}{5R} - \mathcal{E}\,, \tag{11}$$

with respect to the parameter $\mathcal{E}$ used in the reduction (7). This treatment is only valid, of course, to the extent that Diósi's approximation to a harmonic potential is valid, i.e. for widths of the ground state that are quite smaller than the radius R of the sphere. As is established in [11], a static solitonic solution of gaussian shape exists in this limit,

$$\varphi(\mathbf{x}) = \frac{1}{(\sqrt{\pi}A)^{3/2}} \exp\big(-\frac{|\mathbf{x}|^2}{2A^2}\big)\,,$$

with width

$$A = (\frac{\hbar^2}{GM^3})^{\frac{1}{4}} R^{\frac{3}{4}}\,. \tag{12}$$

2.3 Factorisation ansatz and boosted solitonic solutions in absence of external potential.

In the first part of this section, we presented three non-linear generalisations of the free linear Schrödinger equation $i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m}$ that can be cast in the form

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m} + V^{NL}(\Psi)\Psi(t,\mathbf{x}), \tag{13}$$

with $V^{NL}(\Psi)$ a potential which non-linearily depends on the wave function Ψ. In each case, we know that there exists a localized solution of the type $\phi^0_{NL}(\mathbf{x})e^{-iE_0t/\hbar}$ (with $\phi^0_{NL}(\mathbf{x}) = \varphi(\mathbf{x})$ and $E_0 = -\hbar\mathcal{E}$)

which behaves as a static bright soliton. Let us now search for new solutions by imposing our ansatz:

$$\Psi(t,\mathbf{x}) = \Psi_L(t,\mathbf{x}) \cdot \phi_{NL}(t,\mathbf{x}) \tag{14}$$

Substituting (14) in (13) we get

$$\begin{aligned} i \quad & \hbar \cdot ((\frac{\partial\Psi_L(t,\mathbf{x})}{\partial t})\phi_{NL}(t,\mathbf{x})) + \Psi_L(t,\mathbf{x}) \cdot (\frac{\partial\phi_{NL}(t,\mathbf{x})}{\partial t})) = \\ - \quad & \frac{\hbar^2}{2m}\Delta\Psi_L(t,\mathbf{x}) \cdot \phi_{NL}(t,\mathbf{x}) \\ - \quad & \frac{\hbar^2}{2m}(2\bigtriangledown\Psi_L(t,\mathbf{x}) \cdot \bigtriangledown\phi_{NL}(t,\mathbf{x}) + \Psi_L(t,\mathbf{x}) \cdot \Delta\phi_{NL}(t,\mathbf{x})) \\ + \quad & V^{NL}(\Psi)\Psi(t,\mathbf{x}), \end{aligned} \tag{15}$$

that we replace by a system of two equations[4], making use of the identity

[4]This replacement is not one to one in the sense that there could exist solutions of

$\bigtriangledown\Psi_L(t,\mathbf{x}) = (\bigtriangledown A_L(t,\mathbf{x}))e^{i\varphi_L(t,\mathbf{x})} + \Psi_L(t,\mathbf{x})i\bigtriangledown\varphi_L(t,\mathbf{x})$:

$$i\hbar\cdot\frac{\partial\Psi_L(t,\mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(t,\mathbf{x}), \tag{16}$$

$$\begin{aligned} i \quad & \hbar\cdot\frac{\partial\phi_{NL}(t,\mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m}\cdot\Delta\phi_{NL}(t,\mathbf{x}) \\ - \quad & \frac{\hbar^2}{2m}\cdot(2i\bigtriangledown\varphi_L(t,\mathbf{x})\cdot\bigtriangledown\phi_{NL}(t,\mathbf{x}) + \frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown\phi_{NL}(t,\mathbf{x})) \\ + \quad & V^{NL}(\Psi)\phi_{NL}(t,\mathbf{x}) \end{aligned} \tag{17}$$

We are free to solve equation (16) by imposing a plane wave solution:

$\Psi_L(t,\mathbf{x}) = A\cdot e^{i\varphi_L(t,\mathbf{x})}$ with $A_L = A$ a positive real constant and $\varphi_L(t,\mathbf{x}) = \mathbf{k}\cdot\mathbf{x} - \omega\cdot t$, with $\omega = \hbar k^2/2m$. The scaling properties of the non-linear potentials considered by us are such that $V^{NL}(\Psi) = V^{NL}(\Psi_L(t,\mathbf{x})\cdot\phi_{NL}(t,\mathbf{x})) = A^2V^{NL}(\phi_{NL}(t,\mathbf{x}))$.

Let us rescale $\phi_{NL}(t,\mathbf{x})$ by imposing that $\phi_{NL}(t,\mathbf{x}) = \phi'_{NL}(t,\mathbf{x})/A$; then, $V^{NL}(\Psi) = V^{NL}(\phi'_{NL}(t,\mathbf{x}))$, so that $\phi'_{NL}(t,\mathbf{x})$ must fulfill the equation

$$\begin{aligned} i\hbar\cdot\frac{\partial\phi'_{NL}(t,\mathbf{x})}{\partial t} &= \\ -\frac{\hbar^2}{2m}\cdot\Delta\phi'_{NL}(t,\mathbf{x}) - i\frac{\hbar^2}{m}\bigtriangledown\varphi_L(t,\mathbf{x})\cdot\bigtriangledown\phi'_{NL}(t,\mathbf{x}) & \\ +V^{NL}(\phi'_{NL}(t,\mathbf{x}))\phi'_{NL}(t,\mathbf{x}), & \end{aligned} \tag{18}$$

for which it is straightforward to check that there exists a solution of the type $\phi^0_{NL}(\mathbf{x}-\mathbf{v}\cdot t)e^{-iE_0t/\hbar}$, with $\mathbf{v} = \hbar\bigtriangledown\varphi_L(t,\mathbf{x})/m = \hbar\mathbf{k}/m$, in accordance with de Broglie's relation.

Putting all these results together, we find a solution $\phi^0_{NL}(\mathbf{x}-\mathbf{v}\cdot t)e^{-i((E_0+\hbar\omega)\cdot t-\hbar\mathbf{k}\cdot\mathbf{x})/\hbar}$. Actually, this class of solution is well-known and it can be generated from the static solution $\phi^0_{NL}(\mathbf{x})e^{-iE_0t/\hbar}$ by a Galilean boost. This result is in a sense trivial because we considered to begin with equations which are Galilei invariant. In the next section, we shall generalize this idea by realizing a time-dependent Galilean boost, in accordance with a generalized dB-B guidance equation.

equation (15) that do not fulfill the system (16,17). In any case, we focus on a particular class of solutions here. Our goal is not to solve (19) for arbitrary initial conditions. We actually assume that to begin with the full wave satisfies our factorisability ansatz.

3 In presence of an external potential.

We now assume that an external, linear, potential acts on the particle and we must replace (13) by

$$i\hbar\frac{\partial\Psi(t,\mathbf{x})}{\partial t} = -\hbar^2\frac{\Delta\Psi(t,\mathbf{x})}{2m} + V^L(t,\mathbf{x})\Psi(t,\mathbf{x}) + V^{NL}(\Psi)\Psi(t,\mathbf{x}). \quad (19)$$

By repeating the scheme already applied in the previous section we derive a system of two coupled equations, the linear Schrödinger equation

$$i\hbar\cdot\frac{\partial\Psi_L(t,\mathbf{x})}{\partial t} = -\frac{\hbar^2}{2m}\Delta\Psi_L(t,\mathbf{x}) + V^L(t,\mathbf{x})\Psi_L(t,\mathbf{x}), \quad (20)$$

and as before equation (17).

In order to solve the system of equations (16,17), it is worth noting that while the L_2 norm of the linear wave Ψ_L is preserved throughout time, this is no longer true in the case of the non-linear wave ϕ_{NL}, because the interaction terms coming from the Laplacian operator are not hermitian.

By a straightforward but lengthy computation that we reproduce in appendix, we were able to establish the following results:

- the solution of (17) obeys the scaling law

$$\frac{\frac{d<\phi_{NL}|\phi_{NL}>}{dt}}{<\phi_{NL}|\phi_{NL}>} = -2\frac{\frac{dA_L}{dt}}{A_L}, \quad (21)$$

  in the limit where A_L and ϕ_L vary smoothly.

- In equation (21) we introduced the total derivative $\frac{dA_L}{dt}$ defined through

$$\frac{dA_L}{dt} = \frac{\partial A_L}{\partial t} + \mathbf{v}_{drift}\cdot\bigtriangledown A_L, \quad (22)$$

  where $\mathbf{v}_{drift}$ is in turn defined as follows:

$$\mathbf{v}_{drift} = \frac{\frac{d<\phi_{NL}|\mathbf{x}|\phi_{NL}>}{dt}}{<\phi_{NL}|\phi_{NL}>}, \quad (23)$$

  for which we found by direct computation the generalisation of the dB-B guidance equation:

$$\mathbf{v}_{drift} = \frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t) + \frac{<\phi_{NL}|\frac{\hbar}{im}\nabla|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$$
$$= \mathbf{v}_{dB-B} + \mathbf{v}_{int.}, \qquad (24)$$

in which $\mathbf{v}_{int.}$ can be considered as a contribution to the average velocity originating from the internal structure of the soliton. The dB-B contribution to the drift, $\frac{\hbar}{m}\nabla\varphi_L(\mathbf{x_0}(t),t)$, denoted $\mathbf{v}_{dB-B}$ is evaluated at the barycentre of the soliton, from now on denoted $\mathbf{x_0}$. In the rest of the paper, we shall refer to the guidance equation (24) as the "Disturbed" dB-B guidance equation.

- $\mathbf{v}_{dB-B}$ appears in (24) as a consequence of the coupling to φ_L (as in the free case treated in section 2.3), through the term $-i\frac{\hbar^2}{m}\nabla\varphi_L(t,\mathbf{x})\cdot\nabla\phi_{NL}(t,\mathbf{x})$ present in the Hamiltonian of equation (17), that we shall from now on call the guidance potential.
- The coupling to A_L (absent in the free case treated in section 2.3), that is to say, the term $-\frac{\hbar^2}{m}\frac{\nabla A_L(t,\mathbf{x})}{A_L}\cdot\nabla\phi_{NL}(t,\mathbf{x})$ present in the Hamiltonian of equation (17), that we shall from now on call the $A_L - \phi_{NL}$ potential, does not contribute directly to the drift velocity but as we show in appendix it contributes to the scaling. It also contributes indirectly to the drift velocity because it influences the shape of ϕ_{NL}, and thus also influences $\mathbf{v}_{int.}$.
- From the constraint (21) we infer that $\frac{<\phi_{NL}|\phi_{NL}>(t)}{<\phi_{NL}|\phi_{NL}>(t=0)} = \frac{A_L^2(t=0)}{A_L^2(t)}$, where we evaluate $A_L^2(t)$ at the barycentre of ϕ_{NL}, which moves according to the Disturbed dB-B guidance equation (24).

In analogy with the rescaling performed in the free case, let us define ϕ'_{NL} through

$$\phi_{NL}(t,\mathbf{x}) = \frac{1}{A_L(t,\mathbf{x_0})}\cdot\phi'_{NL}(t,\mathbf{x}). \qquad (25)$$

(17) can the be cast in the form

$$\begin{aligned}
i\hbar \quad & \cdot\frac{\partial(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0}))}{\partial t} \\
= \quad & -\frac{\hbar^2}{2m}\cdot\Delta(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) \\
- \quad & \frac{\hbar^2}{m}\cdot i\bigtriangledown\varphi_L(t,\mathbf{x})\cdot\bigtriangledown(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) \\
- \quad & \frac{\hbar^2}{m}\cdot\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})) \\
+ \quad & V^{NL}(\phi'_{NL})(\phi'_{NL}(t,\mathbf{x})/A_L(t,\mathbf{x_0})).
\end{aligned} \tag{26}$$

We know from (21) that in very good approximation, the L_2 norm of $\phi'_{NL}(t,\mathbf{x})$ remains constant. If we consider an extreme "adiabatic" limit in which the logarithmic gradient of A_L, $\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}$ can be neglected, we find that the soliton propagates like a classical particle, because then it is also consistent to perform a WKB like treatment in which ϕ_L is proportional to S, the solution of the classical Hamilton-Jacobi equation[5].

Without being so extreme, we are in right to claim that, if $(1/A_L(t,\mathbf{x_0}(\mathbf{t})))$ varies smoothly enough so that in first approximation we may neglect its temporal derivative in (26), as well as the $A_L-\phi_{NL}$ coupling, then, $1/A_L(t,\mathbf{x_0}(\mathbf{t}))$ factorizes in (26). If it is so, $\phi'_{NL}(t,\mathbf{x})$ obeys (18) for which we find[6], as announced in the introduction, that the solution is a progressive solitary wave

$$\Psi(x,y,z,t) \approx e^{-iE_0t/\hbar}\phi^0_{NL}(\mathbf{x}-\int_0^t dt\mathbf{v}_{dB-B})e^{i\varphi_L(\mathbf{x},t)}, \tag{27}$$

where $\mathbf{v}_{dB-B} = \frac{\hbar}{m}\nabla\varphi_L(x,y,z,t)$, in accordance with the dB-B guidance equation, and E_0 is the energy of the stationary ground state of the free non-linear equation.

This is a crude approximation however, and all we can predict in general is that the solution, when it exists and remains peaked

[5]In the Bohmian language this occurs when we can consistently neglect the so-called quantum potential [23].

[6]This approximation is a good approximation in the regime where we may neglect the variation of φ_L over the size of the soliton. Consistently, in this limit, $\mathbf{v}_{int.} = 0$ because ϕ^0_{NL} is a static solution of the free non-linear equation.

throughout time, has the form

$$\Psi(x, y, z, t) \approx \phi'_{NL}(\mathbf{x}, t) e^{i\varphi_L}, \tag{28}$$

where $\phi'_{NL}(\mathbf{x}, t)$'s norm is quasi-constant, while its barycentre is located in $\mathbf{x}_0(t = 0) + \int_0^t dt \mathbf{v}_{drift}$. In order to say more about $\phi'_{NL}(\mathbf{x}, t)$ we must solve equation (26) which is a complicated problem, beyond the scope of our paper. It is worth noting however that if we estimate $\mathbf{v}_{int.}$ at the first order of linear perturbation theory, we find identically 0, because this contribution is proportional to the average velocity of the free soliton, evaluated in the frame where it is at rest.

What remains to do in order to find better approximations of the exact solution of (17) is to resort to perturbative methods. This is appropriate, having in mind, that the L_2 norm of ϕ'_{NL} does not vary much and that the non-linear potential "seen" by ϕ'_{NL} does not rescale, which opens the door to a perturbative approach.

4 A typical scattering experiment.

Let us consider a typical scattering experiment, where a beam of identical particles, all with the same velocity and direction, are sent on a target. In the usual, linear formulation of quantum mechanics, the position of the particle is randomly distributed according to the Born rule. Moreover, before it reaches the target, the beam is described by a linear plane wave (we neglect here size effects). According to the Born rule everything happens as if the particle was uniformly spread over the beam.

Making use of the results of the previous sections we are free to adopt another picture in which the particle is attached to a peaked soliton, maybe since eons. In this picture a particle consists of a very dense concentration of field which is thus always well-localized. As discussed in section2.3, the particle (soliton) moves at constant speed in the direction of the target. This situation was actually considered by de Broglie, who also assumed that inside the beam the probability of localisation of the particles/singularities/second solutions and so on was homogeneous, due to the symmetry of the beam[7]. In this case, the predictions made in standard, linear quantum mechanics and in

[7]This argument of symmetry implicitly refers to the existence of what is called nowadays an equilibrium distribution for the hidden positions of the particles. There exist serious attempts to derive the onset of quantum equilibrium from the de Broglie-Bohm mechanics

the non-linear model outlined above are the same. Now, because the equilibrium distribution in $|\Psi_L|^2$ is, as is well-known, equivariant under de Broglie-Bohm mechanics, the non-linear dynamics sketched in the previous sections also leads to the same predictions for all times, because equilibrium is preserved during the scattering process. Actually, in the non-relativistic regime, this is so in first approximation only, in the regime where the Disturbed dB-B trajectories (24) cannot be distinguished from the dB-B trajectories (1), but we shall now show that if we consider the relativistic Dirac equation, the dB-B guidance equation linked to Dirac's equation is exactly satisfied [33, 23].

5 The Dirac spinor.

We mentioned in the introduction that in our view the NLS and S-N equations are good candidates for fulfilling the de Broglie double solution program of 1927, in the same sense that Schrödinger's equation was a good candidate for realizing de Broglie's wave mechanics program of 1925. In the same line of thought, it is natural to look for a non-linear relativistic equation that would be to the NLS and S-N equations what are the Klein-Gordon or Dirac equations to the non-relativistic Schrödinger equation.

A way to fulfill this program consists of treating the Dirac spinor more or less in the same way as the scalar Schrödinger wave, imposing now the ansatz

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0(t,\mathbf{x}) \\ \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \Psi_0^L(t,\mathbf{x}) \\ \Psi_1^L(t,\mathbf{x}) \\ \Psi_2^L(t,\mathbf{x}) \\ \Psi_3^L(t,\mathbf{x}) \end{pmatrix} \cdot \phi_{NL}(t,\mathbf{x}), \tag{29}$$

where $\phi_{NL}(t,\mathbf{x})$ is a Lorentz scalar. Remarkably, we now find that the equation obeyed by $\phi_{NL}(t,\mathbf{x})$ can be cast in the simple form

$$\frac{i\hbar\partial\phi_{NL}(t,\mathbf{x})}{\partial t} = \mathbf{v_{Dirac}}(t,\mathbf{x})\frac{\hbar}{i}\bigtriangledown\phi_{NL}(t,\mathbf{x}) + V_{NL}(\mathbf{\Psi})\phi_{NL}(t,\mathbf{x}), \tag{30}$$

where $\mathbf{v_{Dirac}}$ can be expressed in terms of the three Dirac α matrices [23] through

[34, 21, 10, 13, 1], but this is a deep and complex problem, reminiscent of the H-theorem of Boltzmann, which opens the door to a Pandora box that we do not wish to open here.

$$\mathbf{v_{Dirac}} = (\Psi_0^*(t,\mathbf{x}), \Psi_1^*(t,\mathbf{x}), \Psi_2^*(t,\mathbf{x}), \Psi_3^*(t,\mathbf{x})) \cdot \alpha c \cdot \begin{pmatrix} \Psi_0(t,\mathbf{x}) \\ \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \end{pmatrix} \quad (31)$$

As before we shall now assume that the self-interaction obeys the usual scaling law $V_{NL}(\lambda\mathbf{\Psi}) = |\lambda|^2 V_{NL}(\mathbf{\Psi})$ and even more:

$V_{NL}(\mathbf{\Psi})$ is a function supposed to depend only on the "density of stuff" is $(\mathbf{\Psi})^\dagger \cdot (\mathbf{\Psi})(t,\mathbf{x}) = A_L^2 |\phi|^2_{NL}(t,\mathbf{x})$, with

$$A_L = \sqrt{|\Psi_0^L(t,\mathbf{x})|^2 + |\Psi_1^L(t,\mathbf{x})|^2 + |\Psi_2^L(t,\mathbf{x})|^2 + |\Psi_3^L(t,\mathbf{x})|^2}. \quad (32)$$

If the linear Hamiltonian is the free Dirac Hamiltonian, and that we impose properly normalised spinorial plane wave solutions, then $\mathbf{v_{Dirac}}$'s components are constant everywhere in space and we again find "boosted" solutions of (30) of the type $\phi_{NL}(t,\mathbf{x}) = e^{-iE_0 t/\hbar}\phi^0_{NL}(\mathbf{x} - \mathbf{v_{Dirac}} \cdot t)$, where ϕ^0_{NL} is a bright static soliton solution of the constraint[8] $V_{NL}(\phi^0_{NL}(t,\mathbf{x})) = E_0 \phi^0_{NL}(t,\mathbf{x})$.

If we assume that the self-interaction of the electron does preserve its norm and does not contribute to its drift, then, by repeating the computations made in appendix, and also by resorting to the conservation equation $\frac{\partial A_L^2}{\partial t} = -div(A_L^2 \cdot \mathbf{v_{Dirac}}(t,\mathbf{x_0}))$ associated to the linear Dirac equation [33], it is straightforward to establish the same scaling law as in the non-relativistic case studied before:
$\frac{\frac{dA_L}{dt}}{A_L} = \frac{-1}{2} \frac{1}{<\phi_{NL}|\phi_{NL}>} \frac{d<\phi_{NL}|\phi_{NL}>}{dt}$

Consequently, we are entitled to look for wave functions of the form

$$\mathbf{\Psi} = \begin{pmatrix} \Psi_0(t,\mathbf{x}) \\ \Psi_1(t,\mathbf{x}) \\ \Psi_2(t,\mathbf{x}) \\ \Psi_3(t,\mathbf{x}) \end{pmatrix} = \frac{1}{A_L} \cdot \begin{pmatrix} \Psi_0^L(t,\mathbf{x}) \\ \Psi_1^L(t,\mathbf{x}) \\ \Psi_2^L(t,\mathbf{x}) \\ \Psi_3^L(t,\mathbf{x}) \end{pmatrix} \cdot \phi'_{NL}(t,\mathbf{x}), \quad (33)$$

for which we know for sure that in very good approximation (as far as the size of the soliton is quite smaller than the size of the linear pilot-wave):

[8] Actually, in this case, even the linear equation admits a solution of arbitrary shape formally derived by imposing $E_0 = V_{NL} = 0$. However plane waves do not exist in nature and in absence of a non-linear self-focusing, dispersion will spread the soliton sooner or later.

- the L_2 norm of $\phi'_{NL}(t, \mathbf{x})$ is constant throughout time.
- the barycentre of ϕ'_{NL} moves along the hydrodynamical flow lines of Dirac's linear equation, at velocity $\mathbf{v_{Dirac}}$.
- $\phi'_{NL}(t, \mathbf{x})$ obeys the equation

$$\frac{i\hbar\partial\frac{\phi'_{NL}(t,\mathbf{x})}{A_L}}{\partial t} = \mathbf{v_{Dirac}}(t, \mathbf{x})\frac{\hbar}{i}\bigtriangledown\frac{\phi'_{NL}(t, \mathbf{x})}{A_L} + V_{NL}(\phi'_{NL}(t, \mathbf{x})), \tag{34}$$

- which opens the door to a perturbative treatment.
- In particular, if A_L and $\mathbf{v^0_{Dirac}}$ are smooth enough, we find at the lowest order of approximation the solution
  $\phi'_{NL}(t, \mathbf{x}) \approx e^{-iE_0t/\hbar} \cdot \phi^0_{NL}(\mathbf{x} - \mathbf{x^0}(t = 0) - \int_0^t dt\mathbf{v^0_{Dirac}})$
  As in the non-relativistic case, the amplitude A_L is an auxiliary function that disappears at the end of the computation of $\mathbf{\Psi}$.

It is beyond the scope of our paper to specify exactly the expression of V_{NL}. If gravity plays a role it is not granted that V_{NL} ought to be Lorentz covariant actually, and we prefer to leave this difficult question as an open problem.

6 Discussion and Conclusions.

There remain many open questions that we do not pretend to solve here. This concerns, as already mentioned, the reformulation of a H-theorem sketched in the section 4, or the existence and stability of the solitonic solutions considered above. Let us now briefly address some other controversial topics which spontaneously arise in the present context.

Normalisation and Born rule.

One could object that in order to fit to the constraints required by our model, in particular in order to ensure that the size of the soliton is quite smaller than the size of the linear wave, the models of non-linear interaction presented by us in the sections 2.1 and 2.2 would require the NLS coupling constant to be huge, or the mass of the particle to be incredibly heavy in the case of the S-N coupling.

However this is not true. We are free to rescale[9] the solitonic solutions without being constrained by the normalisation to unity of the wave function, which is a condition imposed by the Born rule. In our case, the Born rule is not postulated to begin with, it is rather derived from the equilibrium condition. In our eyes, the equilibrium condition ought to be derived from the Disturbed dB-B dynamics as is done in e.g. classical chaos theory [34, 21, 10, 13, 1]. It is well-known for instance from the study of deterministic chaotic systems that the sensitivity to initial conditions is an essential ingredient for generating stochasticity and impredictability. This ingredient is present in the dB-B and Disturbed dB-B dynamics too [19].

Experiments.

The overwhelming majority of experiments [32, 12] proposed so far in order to reveal the existence of intrinsic non-linearities at the quantum level (like e.g. the self-gravity interaction) is a priori doomed to fail, for what concerns our model, because they systematically took for granted that the wave function was normalised to unity[10]. Therefore, new strategies must be adopted in order to reveal whether or not it is illusory to try to simulate quantum mechanics with realistic waves, and whether departures from the linear paradigm are accessible to experimentalists. For instance, departures from the Born rule could result from the replacement of the dB-B (1) by the Disturbed dB-B (24) guidance equation (section 3). Departure from the Born rule can be tested in the lab. so that in principle our model can be falsified, at least in the classical limit.

Even if our model is not relevant, and in the last resort its relevance ought to be confirmed or falsified by experiments, it could appear to be useful as a phenomenological tool. For instance it could be useful in droplets physics where de Broglie like trajectories have been observed [14, 15, 9]. The dynamics outlined here could also be interesting in cold atoms physics [27] where effective non-linear equations of the S-N type properly describe collective excitations of the atomic density [3]. As far as we know, no de Broglie like trajectory has yet been observed during such experiments. In the same order of ideas, it would be highly

[9] The three non-linear equations that we described in the section 2 posses well defined scaling properties. Actually when the size of the static soliton goes to 0, its norm goes to infinity and its energy goes to minus infinity.

[10] We could impose for instance that the size δ_{NL} of the soliton is the Planck length (more or less 10^{-35} meter). Of course the semi-classical gravity model considered in section 2.2 is not supposed to be relevant at this scale, but formally we are free to make such a choice.

interesting to investigate whether de Broglie like trajectories are good tools for describing optical and/or rogue waves [2, 26]. After all the Choquard equation was initially derived in classical astrophysics, and applied to classical plasma physics, and our factorisability ansatz could be applied to classical non-linear wave equations too.

Conclusions-open questions.

The normalisation of the solitonic solutions considered by us is also left entirely open here; we need an argument that fixes the normalisation once for all. It is not clear whether such an argument can be found in existing theories such as the standard model of particles physics.

Related to this, the scaling of the non-linear potential required for localizing the particle is not arbitrary, but exhibits a $|\Psi|^2$ dependence, typical of the Newton and/or Coulomb self-interaction [36].

All these questions are important of course and they deserve to be scrutinized in depth. At this level, our goal is less ambitious. We hope that the rather simple models treated in this paper will convince the reader that de Broglie's ideas were maybe not that much surrealistic [20] and deprived of consistence. Our results indeed reinforces the dB-B picture according to which the particle non-locally and contextually explores its environment thanks to the nearly immaterial tentacles provided by the solution of the linear Schrödinger equation. This picture is not comfortable but it is maybe the price to pay to restore wave monism[11].

Last but not least, our analysis also confirms several prophetic intuitions originally presented by Louis de Broglie during the Solvay conference of 1927 [5].

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[11] An advantage of wave monism is that, contrary to the Bohmian dynamics formulated in terms of material points, wave monism does not violate the No Singularity Principle formulated by Gouesbet [22]. Everything is continuous in our approach, even if very different spatial scales coexist.

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7 Appendix.

7.1 Change of norm

Let us denote H_L the linear part of the the full Hamiltonian in (17). It is not hermitian, so that $<\phi_{NL}|\phi_{NL}>$, the L_2 norm of its solution $\phi_{NL}(t,\mathbf{x}))$ is not constant throughout time. The non-linear potentials considered by us preserve the L_2 norm however. We can thus evaluate the time derivative of $<\phi_{NL}|\phi_{NL}>$ by direct computation, either integrating by parts, or making use of the formula

$$\begin{aligned}&\frac{d<\phi|O|\phi>}{dt}=\\&<\phi|\frac{\partial O}{\partial t}|\phi>+\frac{1}{i\hbar}<\phi|H_L^\dagger O-OH_L|\phi>\\&=<\phi|\frac{\partial O}{\partial t}|\phi>+\frac{1}{i\hbar}(<\phi|[O,Re.H_L]_-|\phi>\\&+\frac{1}{\hbar}(<\phi|[O,Im.H_L]_+|\phi>),\end{aligned}\tag{35}$$

where O is an arbitrary observable, described by a self-adjoint operator, while $Re.H_L$ and $Im.H_L$, the real and imaginary parts of H_L are self-adjoint operators defined through $2\cdot Re.H_L=H_L+H_L^\dagger$ and $2i\cdot Im.H_L=H_L-H_L^\dagger$.

We find by direct computation that

$$\begin{aligned}&Re.(-\frac{\hbar^2}{m}i\nabla\varphi_L(t,\mathbf{x})\cdot\nabla)\\&=(-\frac{\hbar^2}{m}i\nabla\varphi_L(t,\mathbf{x})\cdot\nabla)-(\frac{\hbar^2}{2m}i\Delta\varphi_L(t,\mathbf{x}))\end{aligned}\tag{36}$$

and $Im.(-\frac{\hbar^2}{m}i\nabla\varphi_L(t,\mathbf{x})\cdot\nabla)=(\frac{\hbar^2}{2m}\Delta\varphi_L(t,\mathbf{x}))$.

Therefore the guidance potential contributes to

$\frac{d<\phi_{NL}|\phi_{NL}>}{dt} = \frac{d<\phi_{NL}|1|\phi_{NL}>}{dt}$ by a quantity

$$(<\phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))|\phi_{NL}> \approx (\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x})) < \phi_{NL}|\phi_{NL}>),$$

due to the fact that, over the size of the soliton, $\varphi_L(t,\mathbf{x})$ and its derivatives are supposed to vary so slowly that we can consistently neglect their variation and put them in front of the L_2 integral.

For estimating the contribution of the $A_L - \phi_{NL}$ coupling to $\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$, we integrate by parts and find

$\frac{\hbar^2}{m}\frac{1}{i\hbar}\int d^3\mathbf{x}(\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown(\phi_{NL}(t,\mathbf{x}))^*\phi_{NL}(t,\mathbf{x}) - (\phi_{NL}(t,\mathbf{x}))^*\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot \bigtriangledown\phi_{NL}(t,\mathbf{x}))$.

We now suppose that we are in right to neglect the variation of $\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}$ in the integral above and we find, integrating by parts, a contribution $-2\frac{\bigtriangledown A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x})$

Putting all these results together, we find that

$$\frac{d<\phi_{NL}|\phi_{NL}>}{dt} \approx \frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot <\phi_{NL}|\phi_{NL}>$$
$$-2\frac{\bigtriangledown A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x}). \qquad (37)$$

7.2 Change of position of the barycentre of the soliton.

By similar computations, we are able to estimate the displacement of the barycentre of the soliton. For instance, let us consider its z component:

$z_0 = \frac{<\phi_{NL}|z|\phi_{NL}>}{<\phi_{NL}|\phi_{NL}>}$ and $\frac{dz_0}{dt} = \frac{1}{<\phi_{NL}|\phi_{NL}>}\frac{d<\phi_{NL}|z|\phi_{NL}>}{dt} - \frac{z_0}{<\phi_{NL}|\phi_{NL}>^2}\frac{d<\phi_{NL}|\phi_{NL}>}{dt}$

We find

$$\frac{dz_0}{dt} = \frac{1}{<\phi_{NL}|\phi_{NL}>}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*(\frac{\hbar\bigtriangledown_\mathbf{z}}{m}\cdot\varphi_L(t,\mathbf{x}))\phi_{NL}(t,\mathbf{x})$$
$$+ \quad \frac{1}{<\phi_{NL}|\phi_{NL}>}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown_\mathbf{z}}{mi}\cdot\phi_{NL}(t,\mathbf{x}) + \frac{1}{<\phi_{NL}|\phi_{NL}>} <\phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))\cdot z|\phi$$
$$+ \quad \frac{\hbar}{im}\int d^3\mathbf{x}(\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown(\phi_{NL}(t,\mathbf{x}))^*\cdot z\cdot\phi_{NL}(t,\mathbf{x}) - (\phi_{NL}(t,\mathbf{x}))^*\cdot z\cdot\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown\phi_{NL}(t,\mathbf{x}$$
$$- \quad \frac{z_0}{<\phi_{NL}|\phi_{NL}>^2}\cdot(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot <\phi_{NL}|\phi_{NL}> -2\frac{\bigtriangledown A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\cdot\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}($$

Now, $\frac{\hbar}{im}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\cdot z\cdot\frac{\bigtriangledown A_L(t,\mathbf{x})}{A_L(t,\mathbf{x})}\cdot\bigtriangledown\phi_{NL}(t,\mathbf{x})\approx z_0\frac{\bigtriangledown A_L(t,\mathbf{x_0})}{A_L(t,\mathbf{x_0})}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x}))$,

$< \phi_{NL}|(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x}))\cdot z|\phi_{NL} > \approx z_0\cdot(\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})\cdot < \phi_{NL}|\phi_{NL} >)$
and so on so that finally we find

$$\frac{dz_0}{dt}=\frac{\hbar\bigtriangledown_{\mathbf{z}}}{m}\cdot\varphi_L(t,\mathbf{x_0})+\frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown_{\mathbf{z}}}{mi}\cdot\phi_{NL}(t,\mathbf{x}),\quad(39)$$

which establishes the Disturbed dB-B guidance equation (24).

7.3 Scaling.

Let us, in accordance with (22), introduce the total time derivative of A_L ($\frac{dA_L}{dt}=\frac{\partial A_L}{\partial t}+\mathbf{v}_{drift}\cdot\bigtriangledown A_L$) where $\mathbf{v}_{drift}$ obeys the Disturbed dB-B guidance equation (24) in virtue of which

$\mathbf{v}_{drift}=\frac{\frac{d< \phi_{NL}|\mathbf{x}|\phi_{NL} >}{dt}}{< \phi_{NL}|\phi_{NL} >}=\frac{\hbar\bigtriangledown}{m}\cdot\varphi_L(t,\mathbf{x_0})+\frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x})$.

By a direct computation, we find

$$\frac{\frac{dA_L}{dt}}{A_L}=\frac{\frac{\partial A_L}{\partial t}}{A_L}+\frac{\bigtriangledown A_L}{A_L}\cdot\frac{\hbar\bigtriangledown}{m}\cdot\varphi_L(t,\mathbf{x_0})+\frac{\bigtriangledown A_L}{A_L}\cdot\frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x})\quad(40)$$

Making use of the conservation equation of the linear Schrödinger equation

$\frac{\partial A_L^2}{\partial t}=-div(A^2\frac{\hbar\bigtriangledown}{m}\cdot\varphi_L(t,\mathbf{x_0}))$ we find $\frac{\frac{\partial A_L}{\partial t}}{A_L}+\frac{\bigtriangledown A_L}{A_L}\cdot\frac{\hbar\bigtriangledown}{m}\cdot\varphi_L(t,\mathbf{x_0})=\frac{-1}{2}div(\frac{\hbar\bigtriangledown}{m}\cdot\varphi_L(t,\mathbf{x_0}))$ and we can rewrite (40) as follows:

$$\frac{\frac{dA_L}{dt}}{A_L}=\frac{-1}{2}\frac{\hbar}{m}\Delta\varphi_L(t,\mathbf{x_0})+\frac{\bigtriangledown A_L}{A_L}\cdot\frac{1}{< \phi_{NL}|\phi_{NL} >}\int d^3\mathbf{x}(\phi_{NL}(t,\mathbf{x}))^*\frac{\hbar\bigtriangledown}{mi}\cdot\phi_{NL}(t,\mathbf{x})\quad(41)$$

Making use of (37), we obtain at the end

$\frac{\frac{dA_L}{dt}}{A_L}=\frac{-1}{2}\frac{1}{< \phi_{NL}|\phi_{NL} >}\frac{d< \phi_{NL}|\phi_{NL} >}{dt}$ which establishes our main result (21).